Multispecies ASEP and t-PushTASEP on a ring and a strange five vertex model

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This talk is an elementary exposition of algebraic and combinatorial aspects of Inhomogeneous t-PushTASEP and Asymmetric Simple Exclusion Process (ASEP) on 1D periodic lattice which become intriguing in **multispecies** setting.

Contents (key words)

Antisymmetric fusion, strange five vertex model, Holstein-Primakov realization of U_q, 3D interpretation of stationary probability, corner transfer matrix, t-oscillator algebra, multiline queue construction, etc.

Based on

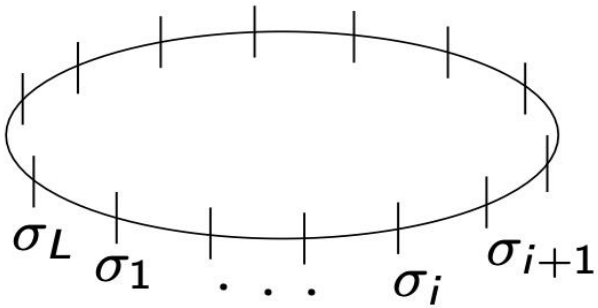
A strange five vertex model and multispecies ASEP on a ring,
K-Okado-Scrimshaw, arXiv:2408.12092,

Multispecies inhomogeneous t-PushTASEP from antisymmetric fusion,
Ayyer-K, arXiv:2503.00829

Inhomogeneous n-Species t-PushTASEP on a ring

[Ayyer-Martin-Williams 2024]

[Corwin-Petrov13, Borodin-Wheeler22, Ayyer-Martin23, Aggarwal-Nicoletti-Petrov23,…]



1D periodic chain with L sites

$$\sigma_i \in \{0, 1, \ldots, n\}$$

0: empty site

1,…,n: species (or color) of particles

$$V = \bigoplus_{\alpha=0}^{n} \mathbb{C}|\alpha\rangle \quad \text{space of one particle states}$$

The space of states of t-PushTASEP is the subspace

$$\mathbb{V}(\mathbf{m}) \subset V^{\otimes L} = \bigoplus_{0 \leqslant \sigma_1, \ldots, \sigma_L \leqslant n} \mathbb{C}|\sigma_1, \ldots, \sigma_L\rangle$$

specified by $\mathbf{m} = (m_0, \ldots, m_n)$, where m_i = number of type i particles.

We assume $m_i \geqslant 1$ for all $0 \leqslant i \leqslant n$.

$$K_i = m_0 + \cdots + m_{i-1} \quad (0 \leqslant i \leqslant n)$$

State vector at time s

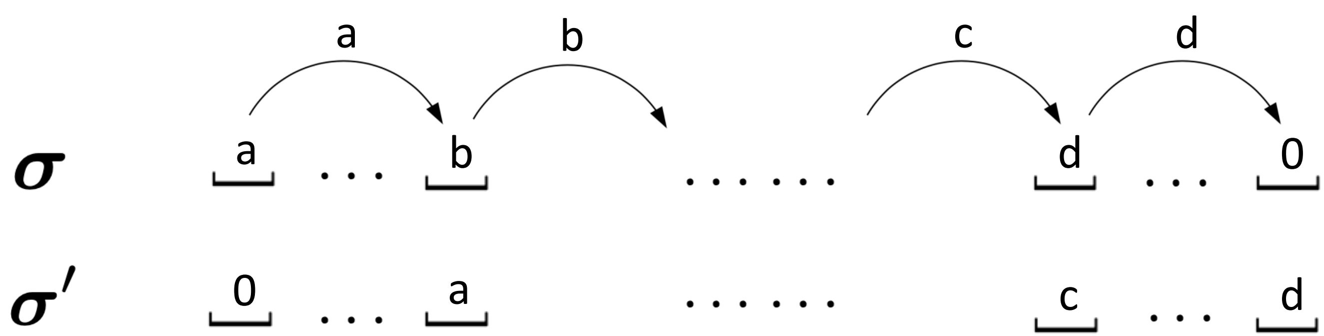
$$|\mathbb{P}(s)\rangle = \sum_{\boldsymbol{\sigma}} \mathbb{P}(\boldsymbol{\sigma}; s)|\boldsymbol{\sigma}\rangle$$

Master equation

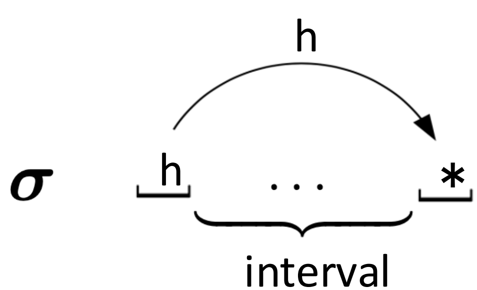
$$\frac{d}{ds}|\mathbb{P}(s)\rangle = H_{\text{PushTASEP}}(x_1, \ldots, x_L)|\mathbb{P}(s)\rangle$$

$$H_{\text{PushTASEP}}(x_1, \ldots, x_L)|\boldsymbol{\sigma}\rangle = \sum_{\boldsymbol{\sigma}'} \sum_{j=1}^{L} \frac{1}{x_j} \prod_{\substack{1 \leqslant h \leqslant n \\ \text{moved}}} w_{\boldsymbol{\sigma},\boldsymbol{\sigma}'}(h)|\boldsymbol{\sigma}'\rangle - \Big(\sum_{j=1}^{L} \frac{[\sigma_j \geqslant 1]}{x_j}\Big)|\boldsymbol{\sigma}\rangle$$

Site j $\qquad$ a > b > ⋯ c > d > 0 $\qquad$ $[\text{true}] = 1, [\text{false}] = 0$



$$\frac{1}{x_j} w_{\boldsymbol{\sigma},\boldsymbol{\sigma}'}(a) \times w_{\boldsymbol{\sigma},\boldsymbol{\sigma}'}(b) \times \cdots \times w_{\boldsymbol{\sigma},\boldsymbol{\sigma}'}(c) \times w_{\boldsymbol{\sigma},\boldsymbol{\sigma}'}(d)$$



$$w_{\boldsymbol{\sigma},\boldsymbol{\sigma}'}(h) = \frac{(1-t)t^{\ell_h}}{1-t^{K_h}}$$

$$\ell_h = \#\{\sigma_i \mid 0 \leqslant \sigma_i < h,\ i \in \text{interval}\}$$

Example. n=2, L=4.

$$H_{\mathrm{PushTASEP}}(x_1, \ldots, x_4)|0121\rangle$$

$$= \frac{|1021\rangle}{x_2} + \frac{|1102\rangle}{(1+t+t^2)x_3} + \frac{t|2101\rangle}{(1+t+t^2)x_3} + \frac{t^2|1201\rangle}{(1+t+t^2)x_3} + \frac{|1120\rangle}{x_4} - \left(\frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right)|0121\rangle$$

Th. [Ayyer-K,2025]

$$H_{\text{PushTASEP}}(x_1,\ldots,x_L) = \frac{1}{(1-t)\prod_{i=1}^{n}(1-t^{K_i})}\sum_{k=0}^{n+1}(-1)^{k-1}\left.\frac{dT^k(z|x_1,\ldots,x_L)}{dz}\right|_{z=0} - \Big(\sum_{j=1}^{L}\frac{1}{x_j}\Big)\mathrm{Id}$$

$T^k(z|x_1,\ldots,x_L)$: commuting transfer matrix of $U_t(\widehat{sl}_{n+1})$ vertex model $(0\leqslant k\leqslant n+1)$

$$[T^k(z|x_1,\ldots,x_L), T^{k'}(z'|x_1,\ldots,x_L)] = 0 \qquad (0\leqslant k,k'\leqslant n+1)$$

z : spectral parameter, x_j : inhomogeneity at site j

auxiliary space = k-th fundamental (degree k antisymmetric tensor) representation V^k $(V^1 = V)$

Individual $T^k(z|x_1,\ldots,x_L)$ and its derivative at $z=0$ are *not* stochastic in general.

A kind of inclusion-exclusion principle that selects sequential particle transitions while canceling unwanted channels.

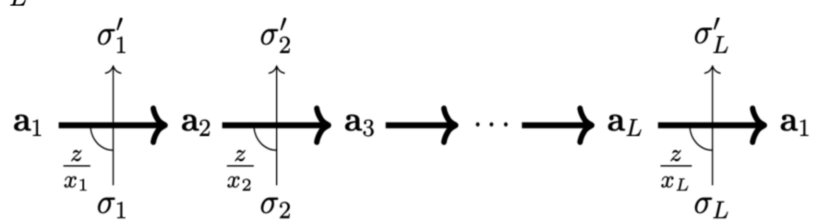
Two natural labeling sets for the base of V^k

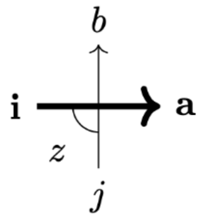
$$\mathcal{T}^k = \{\mathbf{A} = (A_1,\ldots,A_k) \mid 0\leqslant A_1 < \cdots < A_k \leqslant n\} : \text{tableau representation}$$

$$\mathcal{B}^k = \{\mathbf{a} = (a_0,\ldots,a_n) \in \{0,1\}^{n+1} \mid a_0+\cdots+a_n = k\} : \text{multiplicity representation}$$

$$T^k(z|x_1,\ldots,x_L)|\sigma_1,\ldots,\sigma_L\rangle = \sum_{\sigma'_1,\ldots,\sigma'_L} T^k(z|x_1,\ldots,x_L)^{\sigma'_1,\ldots,\sigma'_L}_{\sigma_1,\ldots,\sigma_L}|\sigma'_1,\ldots,\sigma'_L\rangle$$

$$T^k(z|x_1,\ldots,x_L)^{\sigma'_1,\ldots,\sigma'_L}_{\sigma_1,\ldots,\sigma_L} = \sum_{\mathbf{a}_1,\ldots,\mathbf{a}_L\in\mathcal{B}^k}$$





$$= \delta^{\mathbf{a}+\mathbf{e}_b}_{\mathbf{i}+\mathbf{e}_j}(-1)^{a_0+\cdots+a_{j-1}+i_0+\cdots+i_{b-1}} t^{a_{j+1}+\cdots+a_n}(1-t^{a_j}z^{\delta_{b,j}})z^{[j>b]}$$

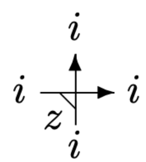
$(\mathbf{e}_j = (0,\ldots,\overset{j}{1},\ldots,0)$: j th elementary vector in $\mathbb{Z}^{n+1})$

This is a special case $(k_1,k_2)=(k,1)$ of the quantum R matrix $S^{k_1,k_2}(z) \in \mathrm{End}(V^{k_1}\otimes V^{k_2})$.

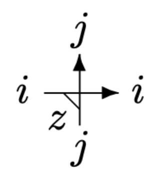
It can be constructed either from 3D L-operator or antisymmetric fusion of $S^{1,1}(z)$.

$S^{1,1}(z)$

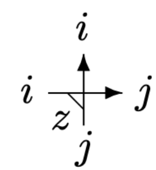
$(i \neq j)$



$1-tz$



$(1-z)t^{[i>j]}$



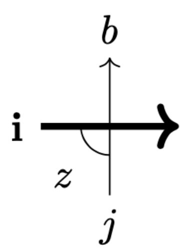
$(1-t)z^{[i<j]}$

singularity

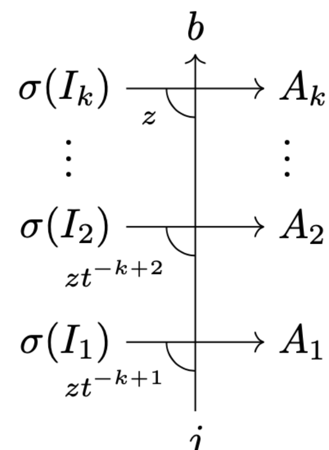
$z=t$: symmetric fusion

$z=t^{-1}$: antisymmetric fusion

$S^{k,1}(z)$



$$= \frac{1}{\prod_{r=1}^{k-1}(1-t^{-r}z)}\sum_{\sigma\in\mathfrak{S}_k}\mathrm{sgn}(\sigma)\times$$



$0 \leqslant b,\ j \leqslant n \qquad \mathbf{i}=(I_1,\ldots,I_k),\ \mathbf{a}=(A_,\ldots,A_k)\in\mathcal{T}^k$ are tableau representations

Example. n=2, L=4.

$$H_{\mathrm{PushTASEP}}(x_1,\ldots,x_4)|0121\rangle = \frac{|1021\rangle}{x_2} + \frac{|1102\rangle}{(1+t+t^2)x_3} + \frac{t|2101\rangle}{(1+t+t^2)x_3} + \frac{t^2|1201\rangle}{(1+t+t^2)x_3} + \frac{|1120\rangle}{x_4} - \Big(\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\Big)|0121\rangle$$

$$H_{\mathrm{PushTASEP}}(x_1,\ldots,x_4) = -\frac{\dot{T}^0(0|x_1,\ldots,x_4) - \dot{T}^1(0|x_1,\ldots,x_4) + \dot{T}^2(0|x_1,\ldots,x_4) - \dot{T}^3(0|x_1,\ldots,x_4)}{(1-t)^2(1-t^3)} - \Big(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\Big)\mathrm{Id}$$

$$\begin{aligned}
T^0(z)|0121\rangle &= \mathcal{D}_0(z)|0121\rangle, \quad T^3(z)|0121\rangle = \mathcal{D}_3(z)|0121\rangle,\\
T^1(z)|0121\rangle &= \frac{(1-t)^4z^2}{x_2x_3}|1012\rangle - \frac{(1-t)^3z(z-x_2)}{x_2x_3}|1102\rangle + \frac{(1-t)^2tz(z-x_1)(tz-x_2)}{x_1x_2x_3}|0112\rangle\\
&+ \frac{(1-t)^2z(z-x_2)(z-x_3)}{x_2x_3x_4}|1120\rangle + \frac{(1-t)^2t^2z(z-x_1)(z-x_4)}{x_1x_3x_4}|0211\rangle - \frac{(1-t)^3tz^2(z-x_4)}{x_2x_3x_4}|2011\rangle\\
&+ \frac{(1-t)^2tz(z-x_2)(z-x_4)}{x_2x_3x_4}|2101\rangle + \frac{(1-t)^2z(z-x_3)(tz-x_4)}{x_2x_3x_4}|1021\rangle + \mathcal{D}_1(z)|0121\rangle,\\
T^2(z)|0121\rangle &= \frac{(1-t)^2tz(tz-x_1)(tz-x_2)}{x_1x_2x_3}|0112\rangle + \frac{(1-t)^4t^2z^2}{x_3x_4}|1210\rangle + \frac{(1-t)^3t^2z^2(tz-x_2)}{x_2x_3x_4}|2110\rangle\\
&+ \frac{(1-t)^2t^3z(z-x_2)(tz-x_3)}{x_2x_3x_4}|1120\rangle + \frac{(1-t)^2t^2z(tz-x_1)(z-x_4)}{x_1x_3x_4}|0211\rangle + \frac{(1-t)^3t^2z(tz-x_4)}{x_3x_4}|1201\rangle\\
&+ \frac{(1-t)^2t^2z(tz-x_2)(tz-x_4)}{x_2x_3x_4}|2101\rangle + \frac{(1-t)^2t^3z(tz-x_3)(tz-x_4)}{x_2x_3x_4}|1021\rangle + \mathcal{D}_2(z)|0121\rangle.
\end{aligned}$$

$$\begin{aligned}
\dot{T}^0(0)|0121\rangle &= \dot{\mathcal{D}}_0(0)|0121\rangle, \quad \dot{T}^3(0)|0121\rangle = \dot{\mathcal{D}}_3(0)|0121\rangle, \qquad (\mathcal{D}_k(z)\text{: coefficient of diagonal term})\\
\dot{T}^1(0)|0121\rangle &= \frac{(1-t)^2}{x_2}|1021\rangle + \frac{(1-t)^2t}{x_3}|0112\rangle + \frac{(1-t)^2t^2}{x_3}|0211\rangle\\
&+ \frac{(1-t)^3}{x_3}|1102\rangle + \frac{(1-t)^2t}{x_3}|2101\rangle + \frac{(1-t)^2}{x_4}|1120\rangle + \dot{\mathcal{D}}_1(0)|0121\rangle,\\
\dot{T}^2(0)|0121\rangle &= \frac{(1-t)^2t^3}{x_2}|1021\rangle + \frac{(1-t)^2t}{x_3}|0112\rangle + \frac{(1-t)^2t^2}{x_3}|0211\rangle\\
&- \frac{(1-t)^3t^2}{x_3}|1201\rangle + \frac{(1-t)^2t^2}{x_3}|2101\rangle + \frac{(1-t)^2t^3}{x_4}|1120\rangle + \dot{\mathcal{D}}_2(0)|0121\rangle.
\end{aligned}$$

Turning to n-Species ASEP

$$H_{\mathrm{ASEP}} = \sum_{i \in \mathbb{Z}_L} \overbrace{1 \otimes \cdots \otimes 1}^{i} \otimes h_{\mathrm{ASEP}} \otimes \overbrace{1 \otimes \cdots \otimes 1}^{L-i-2} \qquad \text{(Markov matrix)}$$

$$h_{\mathrm{ASEP}}(|\alpha\rangle \otimes |\beta\rangle) = (|\beta\rangle \otimes |\alpha\rangle - |\alpha\rangle \otimes |\beta\rangle) t^{[\alpha > \beta]} \qquad \text{(local Markov matrix)}$$

Well-known origin in homogeneous $T^1(z)$

$$H_{\mathrm{ASEP}} = -(1-t)\frac{d}{dz} \log T^1(z|\mathbf{x}=\mathbf{1})\Big|_{z=1} - tL\,\mathrm{Id},$$ (known as ``Baxter type formula")

where $\mathbf{x} = \mathbf{1}$ means the homogeneous specialization $x_1 = \cdots = x_L = 1$.

This is due to the following properties of the basic (non-fusioned) R-matrix at the *Hamiltonian point* $z = 1$:

$$S^{1,1}(1) = (1-t)\mathcal{P}, \quad \mathcal{P}(|\alpha\rangle \otimes |\beta\rangle) = |\beta\rangle \otimes |\alpha\rangle, \quad \mathcal{P}\frac{dS^{1,1}(z)}{dz}\Big|_{z=1} = -h_{\mathrm{ASEP}} - t\,\mathrm{Id}$$

Corollary

$$[T^1(z|\mathbf{x}=\mathbf{1}), T^k(z'|\mathbf{x}=\mathbf{1})] = 0 \quad \text{leads to} \quad [H_{\mathrm{ASEP}}, H_{\mathrm{PushTASEP}}(\mathbf{x}=\mathbf{1})] = 0.$$

ASEP and homogeneous t-PushTASEP share the same eigenstates.

Joint eigenvector of T^k corresponding to the t-Push TASEP stationary state

$$\Lambda^k(z|x_1,\ldots,x_L) = e_{k-1}(t^{K_1},\ldots,t^{K_n})\prod_{j=1}^{L}\left(1-\frac{tz}{x_j}\right) + e_k(t^{K_1},\ldots,t^{K_n})\prod_{j=1}^{L}\left(1-\frac{z}{x_j}\right)$$

$$e_k(z_1,\ldots,z_n) = \sum_{i_1,\ldots,i_n=0,1,\, i_1+\cdots+i_n=k} z_1^{i_1}\cdots z_n^{i_n} \quad \text{(degree } k \text{ elementary symmetric polynomial)}$$

$$H_{\text{PushTASEP}}(x_1,\ldots,x_L) = \frac{1}{(1-t)\prod_{i=1}^{n}(1-t^{K_i})}\frac{d}{dz}\sum_{k=0}^{n+1}(-1)^{k-1}\left(T^k(z|x_1,\ldots,x_L) - \Lambda^k(z|x_1,\ldots,x_L)\right)\Big|_{z=0}$$

We construct the stationary state of t-PushTASEP as a joint eigenstate

$$T^k(z|x_1,\ldots,x_L)|\mathbb{P}_{\text{mp}}\rangle = \Lambda^k(z|x_1,\ldots,x_L)|\mathbb{P}_{\text{mp}}\rangle \qquad (0 \leqslant k \leqslant n+1)$$

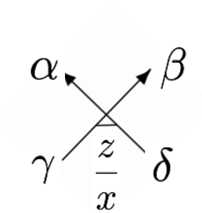
in a matrix product (mp) form

$$|\mathbb{P}_{\text{mp}}\rangle = \sum_{\sigma_1,\ldots,\sigma_L} \mathrm{Tr}\left(A_{\sigma_1}(x_1)\cdots A_{\sigma_L}(x_L)\right)|\sigma_1,\ldots,\sigma_L\rangle \in \mathbb{V}(\mathbf{m})$$

The operators $A_0(x),\ldots,A_n(x)$ are "**corner transfer matrices**" of a **strange five vertex model** (to be explained in detail below).

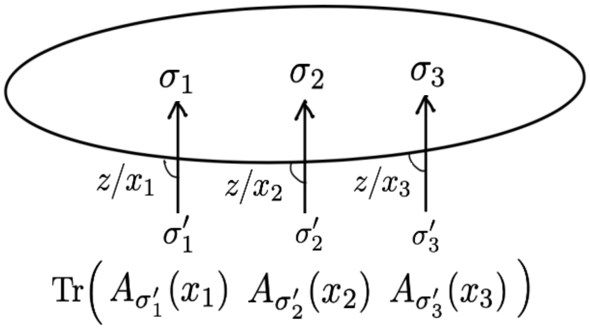
They satisfy the Zamolodchikov-Faddeev(ZF) algebra with structure function $S^{1,1}(z)$.

$$\left(1-\frac{tz}{x}\right)A_\alpha(x)A_\beta(z) = \sum_{\gamma,\delta=0}^{n}$$



$$A_\gamma(z)A_\delta(x)$$

$T^1(z|x_1,\dots,x_L)|\mathbb{P}_{\rm mp}\rangle \overset{?}{=} \Lambda^1(z|x_1,\dots,x_L)|\mathbb{P}_{\rm mp}\rangle$ is depicted as

$\sum_{\sigma'_1,\sigma'_2,\sigma'_3}$ 

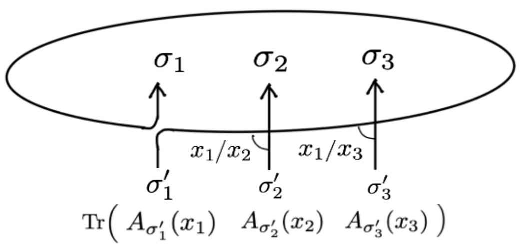
$\overset{?}{=} \Lambda^1(z|x_1,x_2,x_3)\,\mathrm{Tr}\big(A_{\sigma_1}(x_1)A_{\sigma_2}(x_2)A_{\sigma_3}(x_3)\big)$ ----(#)

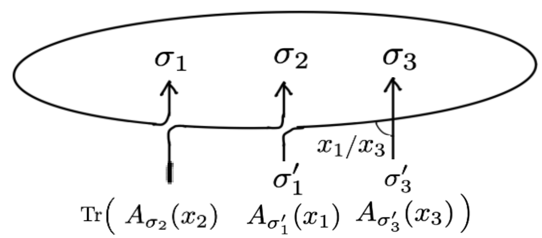
(L=3 example)

(#) is a polynomial equation in z of degree at most L. Suffices to check at $z = 0, x_1, \dots, x_L$.

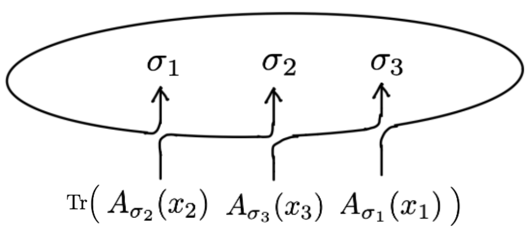
(Crucial advantage of introducing the inhomogeneity.)

$z = 0$ is easy. $z = x_1$ case is shown by successive application of ZF-algebra relation as



× (scalar) =


× (scalar)

= 

× (scalar) $= \Lambda^1(z = x_1|x_1,x_2,x_3)\,\mathrm{Tr}\big(A_{\sigma_1}(x_1)A_{\sigma_2}(x_2)A_{\sigma_3}(x_3)\big)$

factorized

Example of stationary state in $\mathbb{V}(\mathbf{m})$ for $n = 2$

$$\mathbf{m} = (1,1,1): \quad \frac{tx_1 + x_3 + tx_3}{x_1}|012\rangle + \frac{x_2 + x_3 + tx_3}{x_2}|102\rangle + \text{cyc.}$$

$$\mathbf{m} = (1,2,1): \quad \frac{t^2x_1 + x_4 + tx_4 + t^2x_4}{x_1}|0112\rangle + \frac{tx_2 + x_4 + tx_4 + t^2x_4}{x_2}|1012\rangle + \frac{x_3 + x_4 + tx_4 + t^2x_4}{x_3}|1102\rangle + \text{cyc.}$$

$\mathbf{m} = (2,2,1):$

$$\frac{t^2x_1 + t^2x_2 + x_5 + tx_5 + t^2x_5}{x_1x_2}|00112\rangle + \frac{t^2x_1 + tx_3 + x_5 + tx_5 + t^2x_5}{x_1x_3}|01012\rangle + \frac{tx_2 + tx_3 + x_5 + tx_5 + t^2x_5}{x_2x_3}|10012\rangle$$
$$+ \frac{t^2x_1 + x_4 + x_5 + tx_5 + t^2x_5}{x_1x_4}|01102\rangle + \frac{tx_2 + x_4 + x_5 + tx_5 + t^2x_5}{x_2x_4}|10102\rangle + \frac{x_3 + x_4 + x_5 + tx_5 + t^2x_5}{x_3x_4}|11002\rangle + \text{cyc.}$$

cyc. means the terms obtained by cyclic permutation $|\sigma_1, \ldots, \sigma_L\rangle \to |\sigma_L, \ldots, \sigma_{L-1}\rangle, x_i \to x_{i+1}$.

The coefficients are also called "ASEP polynomials", although ASEP stationarity is valid only in the homogeneous specialization $\forall x_i = 1$.

(In that sense, they may better be called "PushTASEP polynomials".)

From now on, we change the label of local states as $0, 1, \ldots, n \to n, \ldots, 1, 0$, and set

$$X_\alpha(z) = A_{n-\alpha}(z^{-1}) \quad (0 \leqslant \alpha \leqslant n)$$

Before explaining its construction by a strange five vertex model, a brief review of relevant results is in order.

Constructions of stationary states of multispecies ASEP on a ring

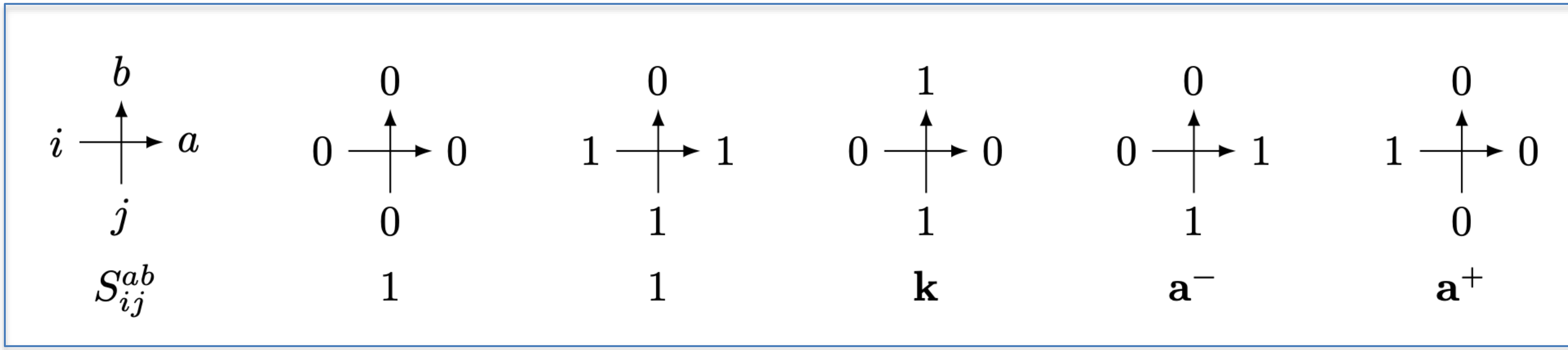
Algebraic	Combinatorial
Matrix product operators	Multiline queue method
Prolhac-Evans-Mallick 2009	$t=0$: Ferrari-Martin (FM) 2009
	$t=t$: Martin 2020
Representations of ZF algebra	(t,q): Corteel-Mandelshtam-Williams 2022
Cantini-de Gier-Wheeler 2015 (Application to Macdonald poly.)	(Application to Macdonald poly.)
$t=0$: ZF alg. from tetrahedron eq.	**$t=0$: FM algorithm from quantum groups**
K-Maruyama-Okado 2016	K-Maruyama-Okado 2015

The key in the KMO approach was a five vertex model whose Boltzmann weights are taken from **t-deformed oscillator algebra at t=0**.

A key for $t \neq 0$ ASEP and t-PushTASEP is yet another t-oscillator five vertex model which obeys a strange weight conservation rule.

It clarifies the relation of the matrix product & multiline queue methods and refines their derivations.

A strange five vertex model



2 state model; a,b,i,j = 0,1. **Strange weight conservation rule a+b=j.**

(cf. Usual weight conservation: $S_{ij}^{ab} = 0$ unless $a + b = i + j$.)

$\mathbf{a}^+, \mathbf{a}^-, \mathbf{k}$ are generators of t-oscillator algebra:

$$\mathbf{k}\,\mathbf{a}^{\pm} = t^{\pm 1}\mathbf{a}^{\pm}\mathbf{k}, \qquad \mathbf{a}^-\mathbf{a}^+ = 1 - t\mathbf{k}, \qquad \mathbf{a}^+\mathbf{a}^- = 1 - \mathbf{k}.$$

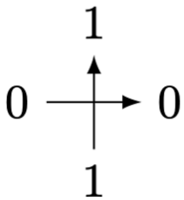
A natural representation on a bosonic Fock space:

$$F := \bigoplus_{d=0}^{\infty} \mathbb{Q}(t)|d\rangle \qquad \mathbf{k}|d\rangle = t^d|d\rangle, \quad \mathbf{a}^+|d\rangle = |d+1\rangle, \quad \mathbf{a}^-|d\rangle = (1-t^d)|d-1\rangle.$$

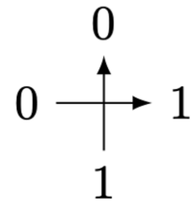
We will also use the number operator $\mathbf{h}$ defined by $\mathbf{h}|d\rangle = d|d\rangle$ so that $\mathbf{k} = t^{\mathbf{h}}$.

(This ket vector is not the one used for an ASEP/t-PushTASEP local state.)

Quantum picture: t-oscillator weighted 2D five vertex model



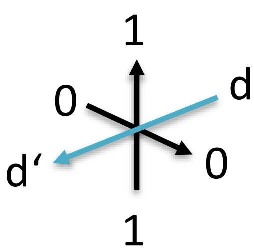
$|d\rangle = t^d |d\rangle$



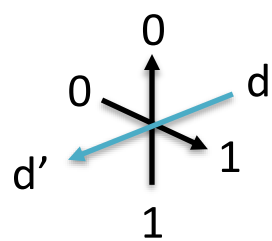
$|d\rangle = (1 - t^d)|d-1\rangle$

etc

Classical picture: 3D vertex model



$= \delta_{d,d'} t^d$



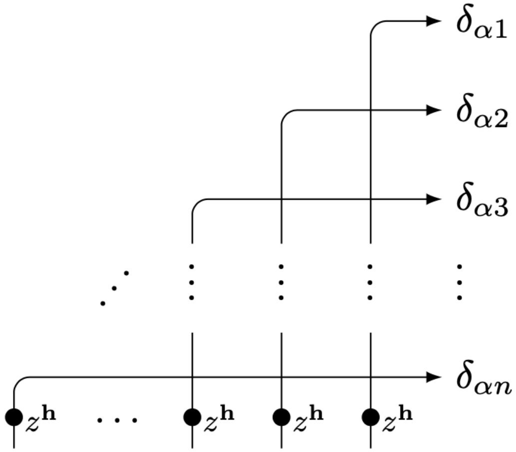
$= \delta_{d-1,d'}(1 - t^d)$

etc

From now on, each 2D vertex i should be understood as carrying an arrow, perpendicular to it, with its own Fock space F running along the arrow, on which a copy of the t-oscillators $\mathbf{k}_i, \mathbf{a}_i^+, \mathbf{a}_i^-$ act.

Define

$X_\alpha(z) =$



= Partition function of the NW quadrant

Boundary condition { Right: fixed; Bottom: free }

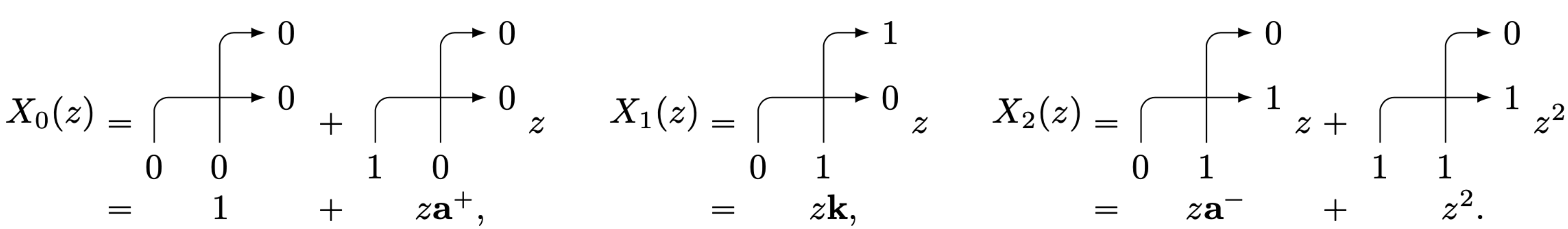
$\bullet = z^{\mathbf{h}} = 1 \text{ or } z \qquad (0 \le \alpha \le n)$

Can be viewed as a **Corner Transfer Matrix(CTM)** (cf. [Baxter, Chap.13]) of the strange five vertex model.

In the classical picture, it is a **layer transfer matrix** of size n for a 3D vertex model defined on a triangular prism.

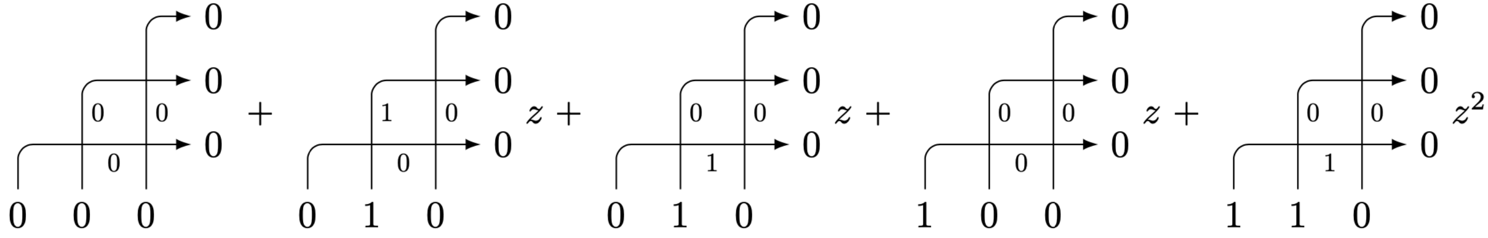
It is a **wiring diagram** for the longest element of the symmetric group S_n.

n=2 case:



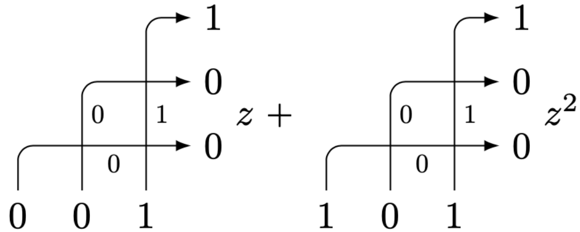
n=3 case:

$X_0(z) =$

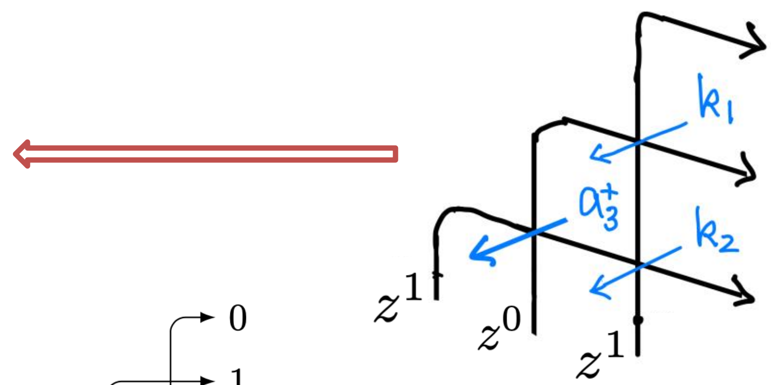


$$= 1 + z\mathbf{a}_1^+\mathbf{k}_3 + z\mathbf{a}_2^+\mathbf{a}_3^- + z\mathbf{a}_3^+ + z^2\mathbf{a}_2^+,$$

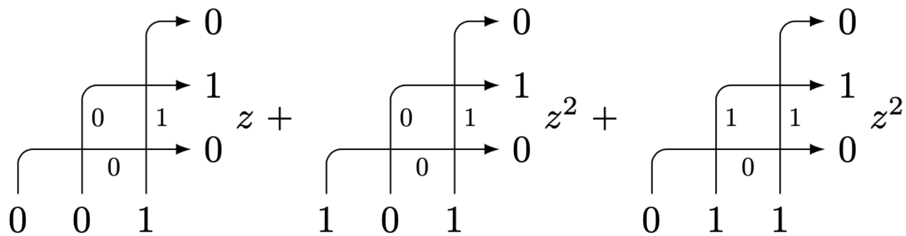
$X_1(z) =$



$$= z\mathbf{k}_1\mathbf{k}_2 + z^2\mathbf{k}_1\mathbf{k}_2\mathbf{a}_3^+,$$

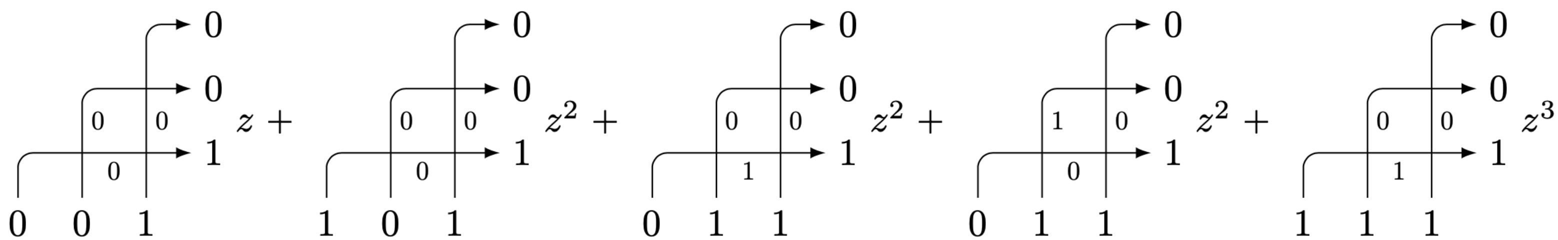


$X_2(z) =$



$$= z\mathbf{a}_1^-\mathbf{k}_2 + z^2\mathbf{a}_1^-\mathbf{k}_2\mathbf{a}_3^+ + z^2\mathbf{k}_2\mathbf{k}_3,$$

$X_3(z) =$



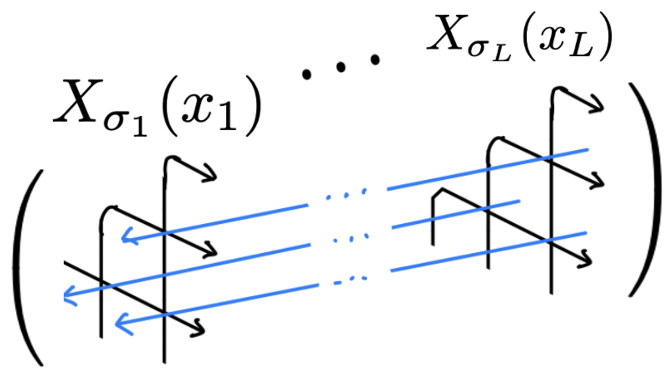
$$= z\mathbf{a}_2^- + z^2\mathbf{a}_2^-\mathbf{a}_3^+ + z^2\mathbf{a}_3^- + z^2\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{k}_3 + z^3.$$

Th. [K-Okado-Scrimshaw 2024] $X_0(z), \ldots, X_n(z)$ satisfy the ZF-algebra relation.

Cor. (Unnormalized) stationary probability for t-PushTASEP on periodic lattice is given by

$$\mathbb{P}(\sigma_1, \ldots, \sigma_L) = \mathrm{Tr}\big(X_{\sigma_1}(x_1) \cdots X_{\sigma_L}(x_L)\big)$$

$$\overset{n=3}{=} \mathrm{Tr}_{F^{\otimes 3}}\Big(\ \Big)$$



= Partition function of a 3D vertex model on a triangular prism whose boundary condition is specified according to $\sigma_1, \ldots, \sigma_L$.

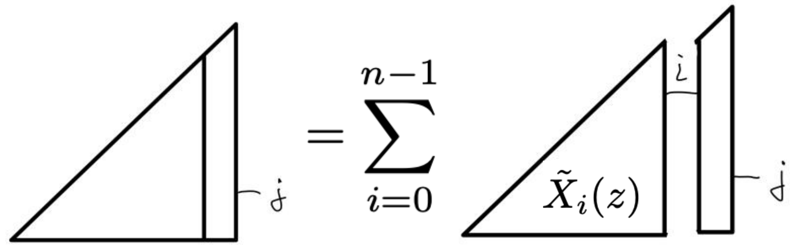
Up to convention, $X_\alpha(z)$ reproduces the one in Cantini–de Gier–Wheeler [CDG 2015], where the ZF-algebra was shown by combining a couple of lemmas.

The diagram rep. for $X_\alpha(z)$ based on the five vertex model here is the simplest one devised to date. (Requires only 0,1 states, whereas [CDG] needs an n-color pen.)

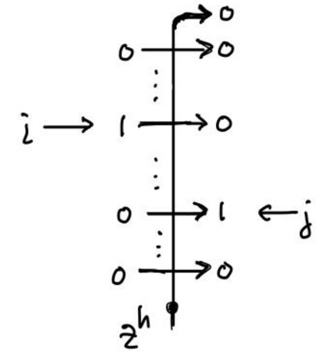
The next page presents key ingredients of our proof, which make use of the CTM diagram and elucidate an connection to the quantum group theory.

n-reducing recursion relation = immediate consequence of the CTM diagram

$$X_j(z) = \sum_{i=0}^{n-1} \tilde{X}_i(z)\ \mathcal{T}(z)_{ij}$$



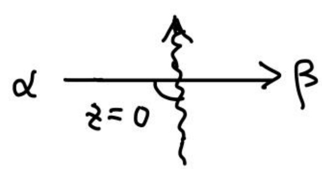
$\mathcal{T}(z)_{ij} =$



The factor $\mathcal{T}(z)_{ij}$ is linked with the quantum group theory via

$$\mathcal{L}_{\alpha,\beta} = \mathcal{T}(z)_{\alpha,\beta+1}(\mathbf{a}_n^-)^{\delta_{\beta n}}(z^{-1}\mathbf{k}_n)^{\theta(\beta\neq n)} \quad (0 \leqslant \alpha \leqslant n-1,\ 0 \leqslant \beta \leqslant n),$$

$\mathcal{L}_{\alpha,\beta} =$



$$= \begin{cases} \mathbf{k}_{\beta+1}\cdots\mathbf{k}_n & (\alpha=\beta) \\ \mathbf{a}_\alpha^+\mathbf{a}_\beta^-\mathbf{k}_{\beta+1}\cdots\mathbf{k}_n & (\alpha<\beta) \\ 0 & (\alpha>\beta) \end{cases} \qquad (0 \le \alpha, \beta \le n)$$

This $\mathcal{L}_{\alpha,\beta}$ is an R-matrix of $U_t(\widehat{sl}_{n+1})$ for an oscillator type representation on $\mathbb{C}^{n+1}\otimes F^{\otimes n}$, which dates back to [Holstein-Primakov 1940].

Relation to multiline queue construction

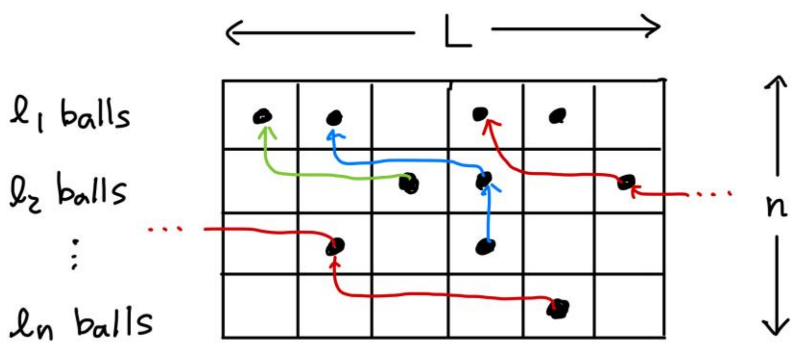
(For simplicity, homogeneous (ASEP) case $x_1 = \cdots = x_L = 1$ from now on.)

Multiline queue (MLQ)
Q = (ball table, pairing)

$\overset{\pi}{\longmapsto} \; W(\mathbf{m})$ (ASEP states)

$\mathbf{m} = (m_0, \ldots, m_n)$

$Q =$



$\longmapsto \; \mathrm{wt}_{q,t}(Q)|\sigma_1(Q), \ldots, \sigma_L(Q)\rangle$

rational function of q, t

$\ell_i = m_i + m_{i+1} + \cdots m_n$

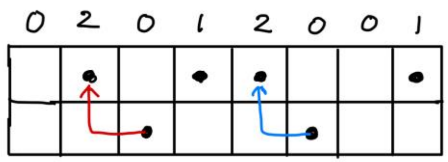
MLQ construction [Martin2020, Corteel-Mandelshtam-Williams2022]

Stationary state in $\mathbb{V}(\mathbf{m})$: $\quad |\mathbb{P}(\mathbf{m})\rangle = \sum_{Q:\mathrm{MLQ}} \pi(Q)|_{q=1}$

Pairing for a given ball table is not unique, and the ASEP state obtained as the image of π carries a coefficient, referred to as the weight.

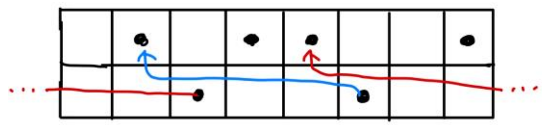
$n = 2$ example $\qquad\qquad \mathbf{m} = (4,2,2) \to (\ell_1, \ell_2) = (4,2)$

$Q_1 =$ ℓ_1 ℓ_2



$$\longmapsto \underbrace{\frac{1-t}{1-qt^4}\,\frac{1-t}{1-qt^3}}_{\mathrm{wt}_{q,t}(Q_1)} |02012001\rangle$$

$Q_2 =$

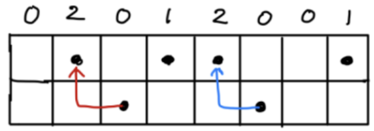


$$\longmapsto \underbrace{\frac{(1-t)qt^2}{1-qt^4}\,\frac{(1-t)t}{1-qt^3}}_{\mathrm{wt}_{q,t}(Q_2)} |02012001\rangle$$

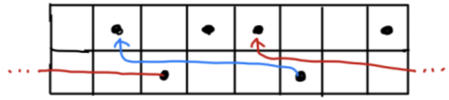
where the weight $\mathrm{wt}_{q,t}(Q)$ of a MLQ is defined combinatorially as

$$\frac{(1-t)^{\ell_2}}{(qt^{\ell_1-\ell_2+1};t)_{\ell_2}} \prod_{\text{pairing line}} t^{\#\{\text{skipped balls}\}} q^{\#\text{wrapping}}$$

($n = 2$ only. For n general, this rule is applied with q replaced by $q^\bullet$ for some $\bullet$.)

$Q_1 =$ 

$$\longmapsto \underbrace{\frac{1-t}{1-qt^4}\,\frac{1-t}{1-qt^3}}_{\mathrm{wt}_{q,t}(Q_1)}|02012001\rangle$$

$Q_2 =$ 

$$\longmapsto \underbrace{\frac{(1-t)qt^2}{1-qt^4}\,\frac{(1-t)t}{1-qt^3}}_{\mathrm{wt}_{q,t}(Q_2)}|02012001\rangle$$

Define

$$M(q,t)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} := \sum_{Q:\mathrm{MLQ}} \mathrm{wt}_{q,t}(Q)$$

vanishing unless a + b = j

= Generating sum of MLQ weights, where dependence on **a, b, i, j** is specified by

Queueing process interpretation

Service

j = balls upstairs =(01011001)

Unused service

b = unconnected balls upstairs =(00010001)

Arriving customers

i = balls downstairs =(00100100)

Used service (Departing customers)

a = connected balls upstairs =(01001000)

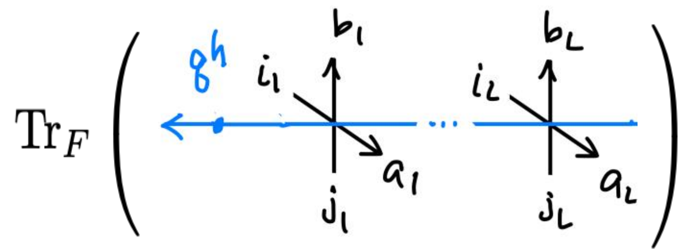
Notation

$$\mathbf{b} = (b_1, \ldots, b_L) \in \{0,1\}^L$$

$$|\mathbf{b}| = b_1 + \cdots + b_L, \quad \text{etc.}$$

Define $$S(q,t)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} := (1-qt^{|\mathbf{j}|-|\mathbf{i}|})\,\mathrm{Tr}\big(q^{\mathbf{h}} S^{a_1 b_1}_{i_1 j_1} \cdots S^{a_L b_L}_{i_L j_L}\big) \qquad (\mathbf{h}|d\rangle = d|d\rangle)$$

= BBQ stick with X shape sausages



This is also vanishing unless **a + b = j** reflecting the strange five vertex model.

Th. [K-Okado-Scrimshaw2024] $M(q,t)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} = S(q,t)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}}$

Example in the previous page:

$$M(q,t)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} = \frac{(1-t)qt^2}{1-qt^4}\frac{(1-t)t}{1-qt^3} + \frac{1-t}{1-qt^4}\frac{1-t}{1-qt^3} = \frac{(1-t)^2(1+qt^3)}{(1-qt^4)(1-qt^3)},$$

$$\begin{aligned} S(q,t)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} &= (1-qt^2)\,\mathrm{Tr}(q^{\mathbf{h}} S^{00}_{00} S^{10}_{01} S^{00}_{10} S^{01}_{01} S^{10}_{01} S^{00}_{10} S^{00}_{00} S^{01}_{01}) \\ &= (1-qt^2)\,\mathrm{Tr}(q^{\mathbf{h}}\mathbf{a}^-\mathbf{a}^+\mathbf{k}\,\mathbf{a}^-\mathbf{a}^+\mathbf{k}) \\ &= (1-qt^2)\sum_{d\ge 0} q^d(1-t^{d+1})t^d(1-t^{d+1})t^d = \frac{(1-t)^2(1+qt^3)}{(1-qt^4)(1-qt^3)}. \end{aligned}$$

A messy sum over the pairings is unified into a single BBQ stick (=Trace) of 5V.
What is 'created' or 'annihilated' by t-oscillator algebra are the customers in the queue.

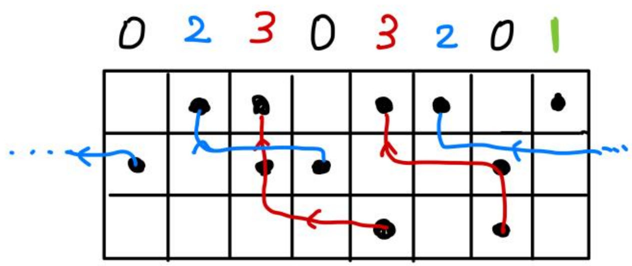
What about n ≥ 3?

n=3 example

MLQ

3-ASEP state

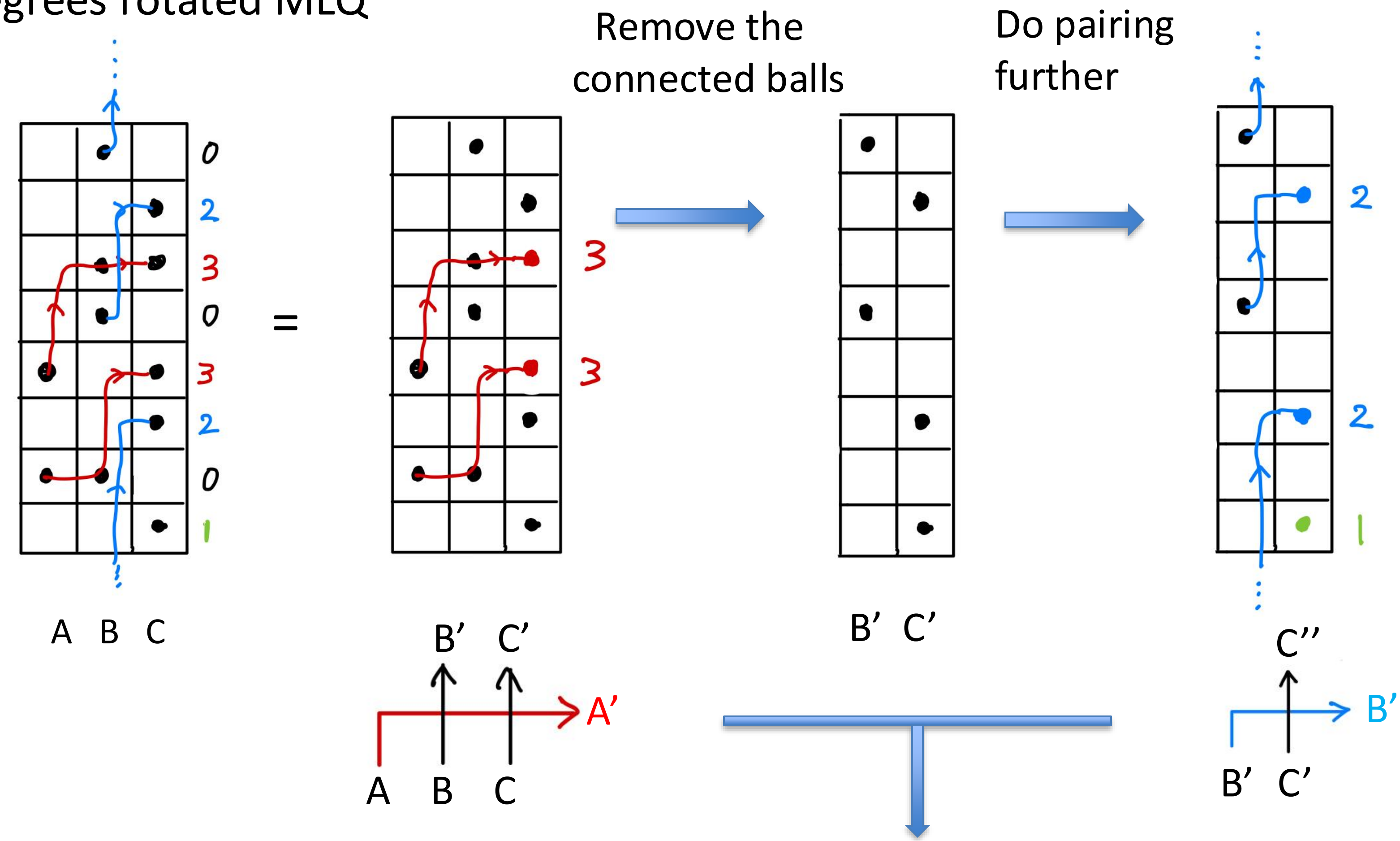
$Q =$



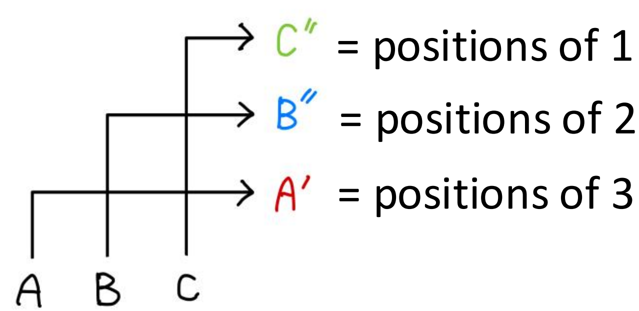
$$\longmapsto \quad \mathrm{wt}_{q,t}(Q)|02303201\rangle$$

The MLQ construction for n ≥ 3 is a nested composition of the n=2 rule in a "CTM manner" as illustrated in the next page.

90 degrees rotated MLQ

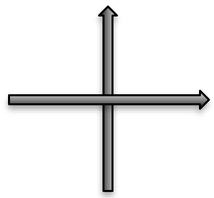


The two diagrams combine into a *single* diagram, which becomes a CTM.

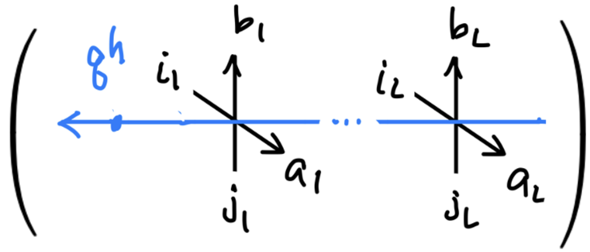


The vertices in the last CTM diagram represent $M(q,t)_{\mathbf{i},\mathbf{j}}^{\mathbf{a},\mathbf{b}}$.

Making the substitution $M(q,t)_{\mathbf{i},\mathbf{j}}^{\mathbf{a},\mathbf{b}} = S(q,t)_{\mathbf{i},\mathbf{j}}^{\mathbf{a},\mathbf{b}}$ (BBQ stick)



Th =

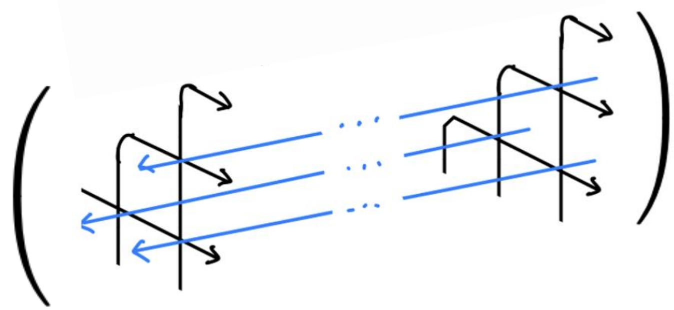
Tr_F 

Vertex encoding MLQ weights

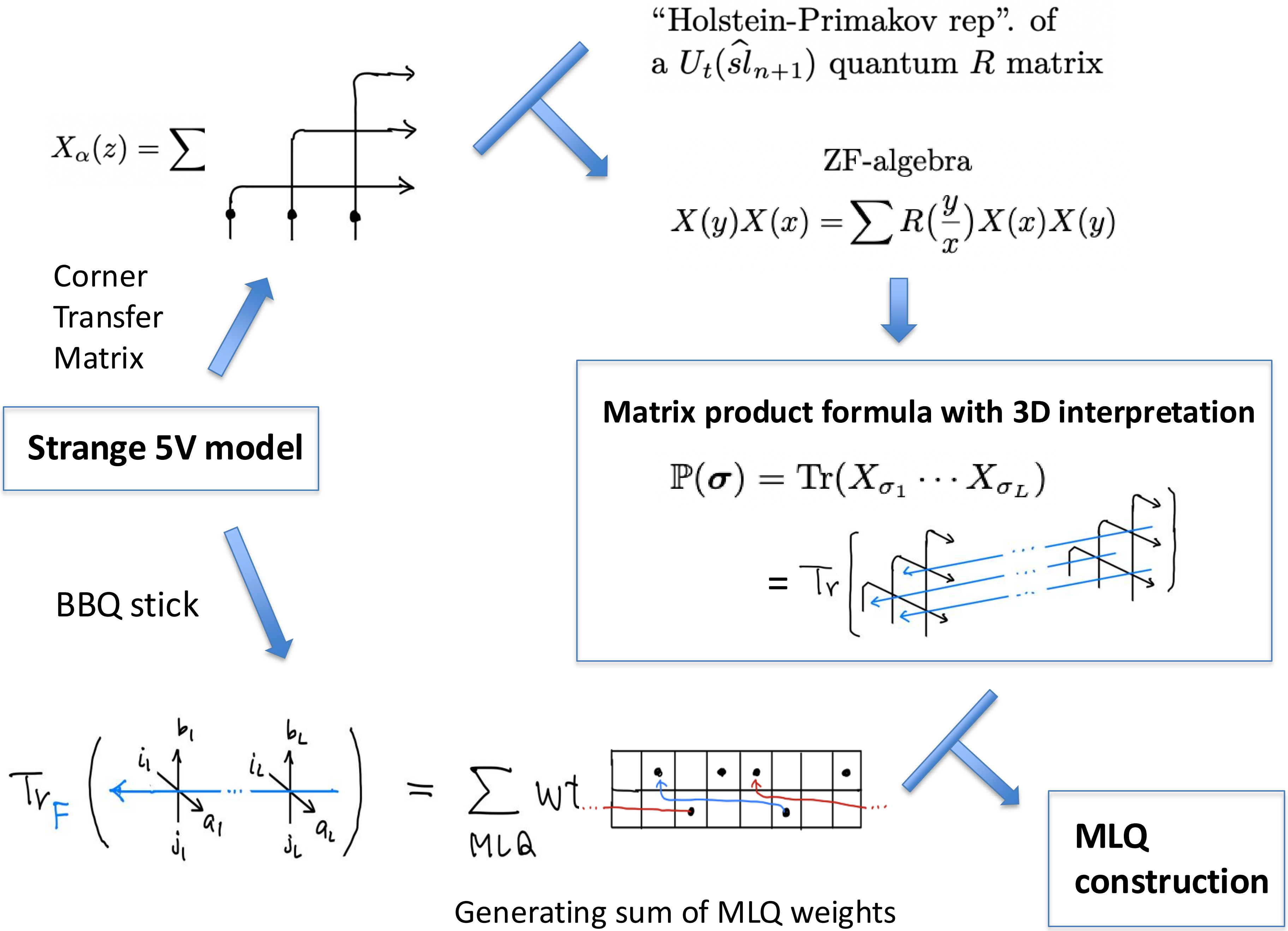
BBQ stick made of the strange five vertex model

and setting $q = 1$, one reproduces the matrix product formula for stationary probabilities, where each layer is a CTM of the strange 5 vertex model ($n = 3$ example shown).

$\mathbb{P}(\sigma_1, \ldots, \sigma_L)$ from MLQ

= Tr 

$$= \mathrm{Tr}(X_{\sigma_1} \cdots X_{\sigma_L})$$


"Holstein-Primakov rep". of
a $U_t(\widehat{sl}_{n+1})$ quantum R matrix
$X_\alpha(z) = \sum$
ZF-algebra
$X(y)X(x) = \sum R(\frac{y}{x})X(x)X(y)$
Corner
Transfer
Matrix
Strange 5V model
Matrix product formula with 3D interpretation
$\mathbb{P}(\boldsymbol{\sigma}) = \mathrm{Tr}(X_{\sigma_1} \cdots X_{\sigma_L})$
= Tr
BBQ stick
Tr_F
b_1
i_1
a_1
j_1
b_L
i_L
a_L
j_L
$= \sum_{MLQ}$ wt
MLQ
construction
Generating sum of MLQ weights

Problems/Questions/Remarks

Superposition over the fundamental representations implies

$$\dim(\text{effective auxiliary space}) = \binom{n+1}{0} + \binom{n+1}{1} + \cdots + \binom{n+1}{n+1} = 2^{n+1}$$

Does this suggest a simpler reformulation of t-PushTASEP?

Are there further examples of commuting transfer matrices that are not stochastic individually, but become so through some form of superposition?
What about type BCD ?

A direct proof of the ZF algebra via a tetrahedron equation (3D analogue of the Yang–Baxter eq.), without relying on induction on n, remains to be found.
(It is known for t=0 [KMO16]).

A further remark

It is also possible to express $H_{\mathrm{PushTASEP}}(x_1, \ldots, x_L)$ in terms of $T_k(z) = T_k(z|x_1, \ldots, x_L)$, which is the commuting transfer matrix whose auxiliary space is the degree k symmetric tensor representation.

$$[T_k(z|x_1, \ldots, x_L), T_{k'}(z'|x_1, \ldots, x_L)] = 0 \qquad (k, k' \in \mathbb{Z}_{\geqslant 0}),$$
$$[T_k(z|x_1, \ldots, x_L), T^l(z'|x_1, \ldots, x_L)] = 0 \qquad (k \in \mathbb{Z}_{\geqslant 0}, l \in \{0, \ldots, n+1\})$$

However, the resulting formula is not particularly illuminating. For example for $n = 2$, one has

$$H_{\mathrm{PushTASEP}}(x_1, \ldots, x_L) = \frac{\dot{T}_2(0) - (1 + t^{m_0} + t^{1+m_0} + t^{m_0+m_1} + t^{1+m_0+m_1})\dot{T}_1(0) + tC\sum_{j=1}^{L}\frac{1}{x_j}}{(1-t)t(1-t^{m_0})(1-t^{m_0+m_1})},$$

$$C = -1 + t - t^{-1+m_0} - t^{2m_0} - t^{1+m_0} - t^{-1+m_0+m_1} - t^{2(m_0+m_1)} - t^{1+m_0+m_1} - t^{-1+2m_0+m_1} - 2t^{2m_0+m_1}$$

where $\dot{T}_l(0) = \left.\frac{dT_l(z)}{dz}\right|_{z=0}$